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CENTERS OF SYMMETRY IN FINITE INTERSECTIONS OF BALLS IN BANACH SPACES

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ABSTRACT
It is proved that a real Banach space X is a G-space (C_σ-space) if and only if the non-empty intersection of three balls with equal radii (any three balls) has a center of symmetry.

Introduction

Intersection properties of balls play an important role in the study of the isometric theory of Banach spaces. In the present paper we shall look at intersection properties of balls from a new point of view. Now we shall pay attention to the symmetry properties of the intersection of a family of balls.

A set S is called centrally symmetric if S has a center of symmetry. The intersection of two balls $B(x,r)$ and $B(y,r)$ with equal radii has a center, the center is $\frac{1}{2}(x+y)$. But if we have three balls with equal radii then their intersection need not have a center. Take for example circular disks in $\mathbf{R}^2$. On the other side the non-empty intersection of three equal squares in $\mathbf{R}^2$ always has a center. In fact it turns out that in the isometric sense the squares are the only balls in $\mathbf{R}^2$ with this property.

In §1 a Banach space is defined to have the SYM(n)-property (E.SYM(n)-property) if the non-empty intersection of n balls with equal radii (any n balls) has a center. We show that a C_σ-space has the E.SYM(n)-property for all n, and that l_1^3 have neither the SYM(3)-property nor the E.SYM(2)-property.

§2 contains the main result. We prove that a real Banach space X has the SYM(3)-property if and only if X is a G-space. This result is a sharpened version of a conjecture of Effros [2]. The proof is given as a sequence of lemmas. The principle of local reflexivity [9] plays an important role in our proof. We use it to

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prove that if X has the SYM(n)-property (the E.SYM(n)-property) then X^{**} has the SYM(n)-property (the E.SYM(n)-property). Lima [4] has characterized spaces without the 3.2.I.P., and spaces with the 3.2.I.P. but without the 4.2.I.P. We use these characterizations to prove that if X has the SYM(3)-property then X has the 4.2.I.P. Hence X^{**} is a $C(K)$-space and we use this in the final step to prove that X is a G-space. As corollaries we get a new characterization of $C(K)$-spaces and the known fact that the range of a norm-1 projection in a G-space is a G-space.

In §3 we examine the E.SYM(2)-property. The intersection of two circular disks in $\mathbf{R}^2$ with different radii need not have a center, but the non-empty intersection of any two squares always has. In fact we have: If $\dim X = n < \infty$ then X has the E.SYM(2)-property if and only if X is isometric to l_∞^n. This finite dimensional result follows easily from the following theorem: If x is a smooth point of the unit ball of a Banach space X with the E.SYM(2)-property and $q \in X^*$ is the unique support functional, then span(q) is an L-summand in X^*.

We do not know in general which spaces are characterized by the E.SYM(2)-property. But if X is a G-space then we are able to prove that X is a C_σ-space. Hence we get that X has the E.SYM(3)-property if and only if X is a C_σ-space. As a corollary we obtain the known result that the range of a norm-1 projection in a C_σ-space is a C_σ-space.

1. Definitions, notation and preliminary results

NOTATION. $B(x,r)$ denotes the closed ball with center x and radius r. The unit ball $B(0,1)$ of a Banach space X will sometimes be written as X_1. We denote the dual space by X^*, and the bidual by X^{**}. By the w*-topology of X^{**} we mean the $\sigma(X^{**}, X^*)$-topology.

If $x \in X$ then $C(x)$ is the smallest facial cone containing x, and if $\|x\| = 1$ then face(x) is the smallest face of X_1 containing x. Combining this we get $C(x) = \text{cone}(\text{face}(x/\|x\|))$.

A Banach space is said to have the n.2.I.P. if every family of n pairwise intersecting balls has a non-empty intersection. See [4] or [8]. The space $H^3(X,(0))$ is the set of all triples (x,y,z) such that $x,y,z \in X$ and $x+y+z=0$, and $\|(x,y,z)\| = \|x\| + \|y\| + \|z\|$. See [4].

DEFINITION. Let S be a subset of a real Banach space X. A point c in X is said to be a center of symmetry of S if $2c - x$ is in S for each x in S

LEMMA 1.1. *A bounded set $S \neq \varnothing$ has a unique center.*

PROOF. Suppose S has two centers, let us say c_1 and c_2. Let $x \in S$. Then $2c_1 - x \in S$ and $y_1 = 2c_2 - (2c_1 - x) = 2(c_2 - c_1) + x \in S$. By repeating this argument we get $y_n = 2n(c_2 - c_1) + x \in S$ for each n. Now $\| y_n - x \| = 2n \| c_2 - c_1 \|$ and S has to be unbounded if $c_1 \neq c_2$.

DEFINITION. A real Banach space X is said to have the SYM(n)-property if the non-empty intersection of n closed balls with equal radii has a center. (It is easy to see that X has the SYM(n)-property if and only if the non-empty intersection of n balls with radii equal to 1 and with $B(0,1)$ as one of the balls, has a center.)

A Banach space X is said to have the E.SYM(n)-property (the extended SYM(n)-property) if the non-empty intersection of any n closed balls has a center.

LEMMA 1.2. *A C_σ-space has the* E.SYM(n)*-property for all n.*

PROOF. Let $C_\sigma(K)$ be the space consisting of all continuous functions on a compact Hausdorff space K which satisfy $f(x) = -f(\sigma x)$ for every $x \in K$, where σ is a homeomorphism of K onto itself whose square is the identity.

Let $\{B(f_i, r_i), i = 1, \cdots, n\}$ be n balls such that

$$S = \bigcap_{i=1}^{n} B(f_i, r_i) \neq \emptyset.$$

Let $g \in C(K)$ be the function defined for each x by

$$2g(x) = \max(f_1(x) - r_1, \cdots, f_n(x) - r_n) + \min(f_1(x) + r_1, \cdots, f_n(x) + r_n).$$

Since $f_i(x) = -f_i(\sigma x)$ for each i we see that $g(x) = -g(\sigma x)$ and hence $g \in C_\sigma(K)$.

Let $h \in S$ and $x \in K$. Then there is an i and a j such that $2g(x) = f_i(x) - r_i + f_j(x) + r_j$. Now $2g(x) - h(x) - f_k(x) = f_i(x) - r_i + f_j(x) + r_j - h(x) - f_k(x) \leqq f_i(x) - r_i + f_k(x) + r_k - h(x) - f_k(x) \leqq r_i - r_i + r_k = r_k$. In a similar way we get $2g(x) - h(x) - f_k(x) \geqq -r_k$. This is true for all x and all k. Thus $2g - h \in S$ and g is a center of S.

LEMMA 1.3. (a) *l_1^3 does not have the* SYM(3)*-property*; (b) *l_1^3 does not have the* E.SYM(2)*-property.*

PROOF. (a) The maximal faces of the unit ball of l_1^3 are triangles. Hence it is easy to arrange three balls with radii equal to 1 such that their intersection is a triangle. Choose for instance $x = (\frac{1}{2}, \frac{1}{2}, 1)$ and $y = (-\frac{1}{2}, 0, \frac{1}{2})$. Then the intersection

$B(0,1) \cap B(x,1) \cap B(y,1)$ will be the triangle with $(\frac{1}{2},0,\frac{1}{2})$, $(0,\frac{1}{2},\frac{1}{2})$ and $(0,0,1)$ as corners. A triangle does not have a center.

(b) Let $x=(1,1,1)$, then $B(0,1) \cap B(x,2)$ is a triangle.

LEMMA 1.4. *The* E.SYM(n)*-property (and the* SYM(n)*-property) is preserved by a norm*-1 *projection.*

PROOF. Suppose X has the E.SYM(n)-property. Let P be a norm-1 projection in X and let $Y=P(X)$ be the range of P. Let B_X denote balls in X and B_Y balls in Y. Let $\{B_Y(y_i,r_i)\}$ be n balls in Y such that

$$S_Y = \bigcap_{i=1}^{n} B_Y(y_i,r_i) \neq \emptyset.$$

Let $S_X = \bigcap_{i=1}^{n} B_X(y_i,r_i)$. Then $S_X \neq \emptyset$ and $P(S_X)=S_Y$. Let c be the center of S_X. Then we easily see that $P(c)$ is the center of S_Y. (In the case X has the SYM(n)-property choose all r_i equal to r.)

2. *G*-spaces and the SYM(3)-property

A Banach space X is called a G-space if the space can be represented as a subspace of some $C(K)$-space consisting of all the functions which satisfy a set A of relations of the form

$$f(x_\alpha)=\lambda_\alpha f(y_\alpha)$$

with $x_\alpha, y_\alpha \in K$ and $|\lambda_\alpha| \leqq 1$ for all $\alpha \in A$. The real G-spaces were introduced by Grothendieck and they have been studied by several authors. See for instance [2, 3, 4, 7, 10, 12, 13, 14, 15, 16].

THEOREM 2.1. *A real Banach space X has the* SYM(3)*-property if and only if X is a G-space.*

The proof will be given as a sequence of lemmas.

LEMMA 2.2. *A G-space has the* SYM(n)*-property for each n.*

PROOF. Let X be a G-space. Then X can be represented as mentioned above. Let $\{B(f_i,r)\}$ be n balls with a non-empty intersection S. The function g defined by $2g=\max(f_1,\cdots,f_n)+\min(f_1,\cdots,f_n)$ is in X by [8, lemma 6.7]. Obviously g is the center of S. (See the proof of Lemma 1.2.) Notice that the center g is independent of the radius r.

LEMMA 2.3. *If X has the* SYM(3)*-property then X^{**} has the* SYM(3)*-property.*

PROOF. Let $x, y \in X^{**}$ such that $S = B(0,1) \cap B(x,1) \cap B(y,1) \neq \emptyset$. If S contains just one point then there is nothing to prove. Hence we may assume that S is infinite. We shall find the center of S as the limit of a net.

By the principle of local reflexivity we may, for each finite subset $\alpha = \{z_1, \cdots, z_n\} \subseteq S$ and for each finite dimensional subspace F in X^*, choose an operator $T_{\alpha,F}$ such that

$$T_{\alpha,F} : E_\alpha \to X,$$

$$(1 - 1/n)\| z \| \leqq \| T_{\alpha,F} z \| \leqq (1 + 1/n)\| z \|,$$

$$T_{\alpha,F} z(f) = f(z),$$

for all $f \in F$ and all $z \in E_\alpha$ where $E_\alpha = \operatorname{span}\{x, y, z_1, \cdots, z_n\}$ and $n = \operatorname{card} \alpha$.

Let now $S_{\alpha,F}$ be the subset of X defined by

$$S_{\alpha,F} = B(0, 1 + 1/n) \cap B(T_{\alpha,F}x, 1 + 1/n) \cap B(T_{\alpha,F}y, 1 + 1/n). \tag{2.1}$$

Let $c_{\alpha,F}$ be the center of $S_{\alpha,F}$. Since $T_{\alpha,F}z_i \in S_{\alpha,F}$ for each i we have $2c_{\alpha,F} - T_{\alpha,F}z_i \in S_{\alpha,F}$. Let $u \in \{0, T_{\alpha,F}x, T_{\alpha,F}y\}$. Then by (2.1)

$$\| 2c_{\alpha,F} - T_{\alpha,F}z_i - u \| \leqq 1 + 1/n. \tag{2.2}$$

Regard now $c_{\alpha,F}$ as a point in X^{**}. Since the net $\{c_{\alpha,F}\}$ is bounded in X^{**} we may assume that the net has a limit c with respect to the w*-topology of X^{**}. (If not choose a convergent subnet.) We may also assume that $\{T_{\alpha,F}x\}$ and $\{T_{\alpha,F}y\}$ are w*-convergent nets in X^{**}, and by the way the operators are chosen we see that

$$T_{\alpha,F}x \to x \quad \text{and} \quad T_{\alpha,F}y \to y. \tag{2.3}$$

Let $z \in S$. Then there is a subnet of $\{T_{\alpha,F}z\}$ w*-converging towards z in X^{**}. By (2.2) and (2.3) we get

$$\| 2c - z \| \leqq 1, \quad \| 2c - z - x \| \leqq 1 \quad \text{and} \quad \| 2c - z - y \| \leqq 1.$$

Hence $2c - z \in S$ and we may conclude that c is the center of S, and consequently X^{**} has the SYM(3)-property.

LEMMA 2.4. *Suppose X has the* SYM(3)*-property. Let $x, y \in X$ and $r > 0$ be such that $S = B(0,r) \cap B(x,r) \cap B(y,r)$ has a non-empty interior. Then the center c of S is an interior point of S.*

PROOF. Let $u \in \operatorname{int} S$. Then there is an $\varepsilon > 0$ such that $B(u, 2\varepsilon) \subseteq S$. Then $2c - B(u, 2\varepsilon) = B(2c - u, 2\varepsilon) \subseteq S$ and hence $B(c, \varepsilon) \subseteq S$.

LEMMA 2.5. *Suppose X has the* SYM(3)*-property. Let $x, y \in X$ and $\tilde{S}$ be the intersection of balls in X^{**} defined by $\tilde{S} = B(0,r) \cap B(x,r) \cap B(y,r)$. Let S be the intersection of balls in X defined by $S = \tilde{S} \cap X$. Suppose r is such that S has a non-empty interior. Then the center c of S is also a center of $\tilde{S}$.*

PROOF. There is by Lemma 2.4 an $\varepsilon > 0$ such that

$$B(c, \varepsilon) \subseteq S. \tag{2.4}$$

Let $a \in \tilde{S}$ and let $E = \text{span}\{x, y, a\}$ in X^{**}. By the principle of local reflexivity we may choose an operator $T_{n,F}$ for each n and each finite dimensional subspace F in X^* such that

$$T = T_{n,F} : E \to X,$$

$$(1-\delta)\|z\| \leqq \|Tz\| \leqq (1+\delta)\|z\|,$$

$$Tz(f) = f(z),$$

for each $z \in E$ and each $f \in F$, and $Tz = z$ for each $z \in X \cap E$ where

$$0 < \delta \leqq \varepsilon / r(n-1). \tag{2.5}$$

Let now $u \in \{0, x, y\}$ and define $b = b_{n,F}$ by

$$b_{n,F} = b = (1/n)c + (1 - 1/n)Ta.$$

By (2.4) and (2.5) we get $\|u - b\| \leqq (1/n)\|u - c\| + (1 - 1/n)\|u - Ta\| \leqq (1/n)(r - \varepsilon) + (1 - 1/n)(1 + \delta)r \leqq r$. Hence $b = b_{n,F} \in S$. Now we may assume that $\{b_{n,F}\}$ is a w*-convergent net in X^{**}, and by the way the operators $T_{n,F}$ are chosen we see that $b_{n,F} \to a$. Since $2c - b_{n,F} \in S \subseteq \tilde{S}$ we get $2c - b_{n,F} \to 2c - a \in \tilde{S}$ since $\tilde{S}$ is w*-closed. We see that c is also a center of $\tilde{S}$, and the center is unique by Lemma 1.1.

LEMMA 2.6. *Suppose X has the* SYM(3)*-property. Let $x, y \in X$ such that $\|x\| = \|y\| = 1$. If* face$(x) \cap$ face$(y) = \varnothing$ *then $\|x - y\| > 1$.*

PROOF. Suppose for contradiction that $\|x - y\| \leqq 1$. Let S be defined by $S = B(0,1) \cap B(x,1) \cap B(y,1)$ and let c be the center of S. Since $0, x, y \in S$ we have $2c$, $2c - x$ and $2c - y \in S$. Then $\|2c\| \leqq 1$, $\|2x - 2c\| \leqq 1$ and $\|2y - 2c\| \leqq 1$. Now $x = \frac{1}{2}(2x - 2c) + \frac{1}{2}2c$ and $y = \frac{1}{2}(2y - 2c) + \frac{1}{2}2c$. Thus we get $2c \in$ face$(x) \cap$ face$(y) = \varnothing$.

LEMMA 2.7. *If X has the* SYM(3)*-property then X has the* 3.2.I.P.

PROOF. It is sufficient to prove that X^* has the 3.2.I.P. (See [4, corollary 3.3].) Suppose X^* does not have the 3.2.I.P. Then there are by [4, corollary 2.11] points $x, y, z \in X^{**}$ such that

$$(x, y, z) \in \partial_e H^3(X^{**}, (0))_1$$

with all $x, y, z \neq 0$. We may assume $\|z\| \leqq \|y\| \leqq \|x\|$. Also $\|z\| = \|x + y\| \leqq \|y\|$ since $x + y + z = 0$. We can now by using [5, lemma 1.1] find u, v, w in X^{**} such that

$$x = w + u, \qquad \|x\| = \|w\| + \|u\|,$$

$$-y = w + v, \qquad \|y\| = \|w\| + \|v\|,$$

$$C(u) \cap C(v) = (0).$$

Use now the fact that $(x, y, z) = (w, -w, 0) + (u, -v, z)$ to write (x, y, z) as a convex combination in $H^3(X^{**}, (0))_1$. But since $z \neq 0$ and (x, y, z) is an extreme point we get $w = 0$. Thus $x = u$ and $-y = v$. Consequently $C(x) \cap C(-y) = (0)$ and

$$\operatorname{face}(x/\|x\|) \cap \operatorname{face}(-y/\|y\|) = \varnothing.$$

Then we get from Lemma 2.3 and Lemma 2.6 that $\|(x/\|x\| + y/\|y\|)\| > 1$. Now by remembering $\|z\| \leqq \|y\| \leqq \|x\|$ and $\|x + y\| = \|z\|$ we get

$$\|x\| \cdot \|y\| < \|(x\|y\| + y\|x\|)\| \leqq \|y\| \cdot \|x + y\| + \|y\| \cdot (\|x\| - \|y\|)$$

$$= \|y\| \cdot \|z\| + \|y\|(\|x\| - \|y\|).$$

Then we get by cancelling $\|y\|$ that $\|x\| < \|z\| + \|x\| - \|y\|$ and we have the contradiction $\|y\| < \|z\| \leqq \|y\|$.

LEMMA 2.8. *If X has the* SYM(3)*-property then X has the* 4.2.I.P.

PROOF. X has the 4.2.I.P. if and only if X^{**} has the 4.2.I.P. Suppose X^{**} does not have the 4.2.I.P. Then there is by [4, corollary 4.5] and Lemma 2.7 above a norm-1 projection P in X^{**} such that $P(X^{**}) = l_1^3$. (We may choose $\varepsilon = 0$ since we are in a dual space.) The SYM(3)-property is preserved by a norm-1 projection (Lemma 1.4), thus contradicting the fact that l_1^3 does not have the SYM(3)-property (Lemma 1.3).

COROLLARY 2.9. *If $\dim X = n < \infty$ then X has the* SYM(3)*-property if and only if X is isometric to l_∞^n.*

PROOF. An n-dimensional space with the 4.2.I.P. is isometric to l_∞^n. The corollary then follows from Lemma 2.2 and Lemma 2.8.

REMARK. Lima [6] has characterized finite dimensional CL-spaces and finite dimensional CL-spaces without the 3.2.I.P. By using these characterizations we could give a more direct and easier proof of Corollary 2.9.

PROOF OF THEOREM 2.1. Lindenstrauss and Wulbert [10, theorem 2] have proved that X is a G-space if and only if for every $x, y \in X$ there is a $u \in X$ such that

$$u(p) = \max(x(p), y(p), 0) + \min(x(p), y(p), 0) \tag{2.6}$$

for every $p \in \partial_e X_1^*$. Let now $x, y \in X$. Choose $r > 0$ such that $S = B(0,r) \cap B(x,r) \cap B(y,r)$ has a non-empty interior in X, and let c be the center of S. We shall show that $2c$ is an element with the property (2.6). By Lemma 2.8 X has the 4.2.I.P., hence X^{**} is a $C(K)$-space with K a compact and extremally disconnected Hausdorff space [8]. Regard now x and y as points in X^{**}. Then z defined by $2z = \max(x, y, 0) + \min(x, y, 0)$ is in $X^{**}(= C(K))$. Let $\tilde{S} = B(0,r) \cap B(x,r) \cap B(y,r)$ in X^{**}, then by Lemma 2.2 we have that z is the center of $\tilde{S}$. Now c is by Lemma 2.5 also a center of $\tilde{S}$ and by the uniqueness of centers (Lemma 1.1) we have $c = z$. Let now $p \in \partial_e X_1^*$. Then $p \in \partial_e X_1^{***}$ and we have

$$2c(p) = 2z(p) = \max(x(p), y(p), 0) + \min(x(p), y(p), 0).$$

The proof is complete.

COROLLARY 2.10. *A real Banach space X is a $C(K)$-space if and only if X has the* SYM(3)*-property and X_1 contains an extreme point.*

PROOF. If the unit ball of a G-space has an extreme point then it is a $C(K)$-space [8].

COROLLARY 2.11 ([10, theorem 3]). *The range of a norm-*1 *projection in a G-space is a G-space.*

PROOF. It follows from Lemma 1.4 and Theorem 2.1.

3. The E.SYM(2)-property

We have not been able to find out which spaces are characterized by the E.SYM(2)-property. If $\dim X < \infty$, then we prove in Theorem 3.6 that they are the l_∞^n-spaces. In the general case, we only have some partial results.

THEOREM 3.1. *Suppose X has the* E.SYM(2)*-property. If x is a smooth point of X_1 and $q \in X^*$ is the support functional, then* span(q) *is an L-summand in X^*.*

We need some lemmas to prove the theorem.

LEMMA 3.2. *If X has the* E.SYM(2)*-property, then X^{**} has the* E.SYM(2)*-property.*

PROOF. The proof is similar to the proof of Lemma 2.3.

Let q be any point in X^* with $\|q\|=1$. Let $D(q)$ be the face of X_1^{**} defined by

$$D(q)=\{f\in X^{**}:\|f\|=1=f(q)\}.$$

Let $H(q)$ be the cone in X^{**} defined by

$$H(q)=\bigcup\{B(nf,n):n\in\mathbf{N}\text{ and } f\in D(q)\}.$$

By the convexity of $D(q)$ it follows that $H(q)$ is convex.

LEMMA 3.3. *If X^{**} has the* E.SYM(2)*-property, then $H(q)$ is* w**-closed.*

PROOF. Let $B(0,1)=X_1^{**}$. We first prove that

$$H(q)\cap B(0,1)=B(0,1)\cap\left(\bigcup\{B(f,1):f\in D(q)\}\right). \tag{3.1}$$

Let $g\in H(q)\cap B(0,1)$. Then there is $f\in D(q)$ and n such that $g\in B(0,1)\cap B(nf,n)=S$. Let h be the center of S. Since $0,f,g\in S$ we have $2h$, $2h-f$ and $2h-g\in S$. Thus $\|2h\|\leqq 1$, $\|2h-g\|\leqq 1$ and $\|1/n((n+1)f-2h)\|\leqq 1$. Now $f=n/(n+1)(1/n((n+1)f-2h))+1/(n+1)2h$. So $2h\in D(q)$ since $D(q)$ is a face, and $g\in B(0,1)\cap B(2h,1)$. Hence we have got the inclusion "$\subseteq$" in (3.1). The converse inclusion is obvious.

Let now $\{g_\alpha\}$ be a w*-convergent net in $H(q)\cap B(0,1)$ such that $g_\alpha\to g_0$. For each α there is by (3.1) an $f_\alpha\in D(q)$ such that $\|f_\alpha-g_\alpha\|\leqq 1$. Let $\{f_\beta\}$ be a w*-convergent subnet of $\{f_\alpha\}$ and let $f_0\in D(q)$ be the limit. Then $\|f_0-g_0\|\leqq 1$, and $g_0\in B(0,1)\cap B(f_0,1)\subseteq B(0,1)\cap H(q)$. Thus $B(0,1)\cap H(q)$ is w*-closed and by Banach–Dieudonné $H(q)$ is w*-closed. The proof of Lemma 3.3 is complete.

Let $F\subseteq X^*$ be defined by

$$F=\{p\in X^*:\|p\|=1\text{ and } f(p)=1\text{ for all } f\in D(q)\}.$$

LEMMA 3.4. *Suppose X^{**} has the* E.SYM(2)*-property. Then $H(q)=\{f\in X^{**}:f(p)\geqq 0$ for all $p\in F\}$.*

PROOF. The inclusion "$\subseteq$" follows directly from the definitions of $H(q)$ and F. Suppose $g\notin H(q)$. Since $H(q)$ is by Lemma 3.3 a w*-closed convex cone there is $p_0\in X^*$, $\|p_0\|=1$ such that $g(p_0)<\inf\{f(p_0):f\in H(q)\}=0$. Let $f\in D(q)$ and let h be such that $h(p_0)=1=\|h\|$. Then $f-h\in B(f,1)\subseteq H(q)$.

Hence $f(p_0)-h(p_0)\geqq 0$, and $1=h(p_0)\leqq f(p_0)\leqq \|p_0\|=1$. Thus $p_0\in F$ and $g(p_0)<0$. The inclusion "$\supseteq$" is proved.

LEMMA 3.5. *There is an* $f_0\in D(q)$ *such that* $B(0,1)\cap H(q)=B(0,1)\cap B(f_0,1)$.

PROOF. We give $D(q)$ the following ordering:

$$f<g \quad \text{if and only if} \quad B(0,1)\cap B(f,1)\subseteq B(0,1)\cap B(g,1).$$

Suppose $f,g\in D(q)$ and $B(0,1)\cap B(f,1)=B(0,1)\cap B(g,1)=S_1$. Then both $f/2$ and $g/2$ are centers of S_1 and by the uniqueness of centers, $f=g$. Thus the ordering is partial.

Let $\{f_\alpha\}$ be a chain in $D(q)$ and let f be a w*-accumulation point in $D(q)$. Let α be given and let $g\in B(0,1)\cap B(f_\alpha,1)$. If $\alpha\leqq\beta$, then $f_\alpha<f_\beta$ and $\|g-f_\beta\|\leqq 1$. Hence $\|f-g\|\leqq 1$ and $g\in B(0,1)\cap B(f,1)$. Thus $f_\alpha<f$ and f is an upper bound of $\{f_\alpha\}$ in $D(q)$.

By Zorn's Lemma, $D(q)$ has a maximal element f_0.

Let $g\in D(q)$. Then $B(0,1)\cap(B(f_0,1)\cup B(g,1))\subseteq B(0,1)\cap B(f_0+g,2)=S_2$. Let h be the center of S_2. Now $S_2\subseteq H(q)$ since $B(f_0+g,2)=B(2(\frac{1}{2}(f_0+g)),2)$. By Lemma 3.4, $f(q)\geqq 0$ for all $f\in S_2$. We have $2h\in S_2$ since $0\in S_2$. Hence $\|2h\|\leqq 1$ and $2h(q)\leqq 1$. Since $f_0\in S_2$ we have $2h-f_0\in S_2$ and $2h(q)\geqq f_0(q)=1$. Thus $\|2h\|=1=2h(q)$ and $2h\in D(q)$.

We also have $S_2\subseteq B(0,1)\cap B(2h,1)$. Thus $f_0<2h$. By the maximality of f_0 and since the ordering is partial we have $f_0=2h$. Hence $B(0,1)\cap B(g,1)\subseteq B(0,1)\cap B(f_0,1)$. Now by (3.1), $H(q)\cap B(0,1)=B(0,1)\cap B(f_0,1)$.

PROOF OF THEOREM 3.1. Let x be a smooth point of X_1 and let $q\in X^*$ be the unique support functional. If we regard x as a point in X^{**} we have $x\in D(q)$ and $F=\{q\}$. Hence $H(q)=\{f\in X^{**}: f(q)\geqq 0\}$ by Lemma 3.4.

Let $J=\{f\in X^{**}: f(q)=0\}$ and let $E=J\cap B(0,1)$. Now we find by using Lemma 3.5 that $E=-f_0+D(q)$, and also $B(0,1)=\{g+tf_0: g\in E \text{ and } |t|\leqq 1\}$. Then we easily get $\|g+tf_0\|=\max\{\|g\|,|t|\}$. Hence $X^{**}=J\oplus_\infty \operatorname{span}(f_0)$. Thus J is an M-summand in X^{**} and so $\operatorname{span}(q)$ is an L-summand in X^*.

THEOREM 3.6. *If* $\dim X=n<\infty$ *then* X *has the* E.SYM(2)*-property if and only if* X *is isometric to* l^n_∞.

PROOF. The if-part follows from Lemma 1.2. We prove the only if-part by induction. The theorem is obviously true for $n=1$. Suppose it is true for $n-1$. Let x be a smooth point of X_1 [11]. Let q be the unique support functional and $Y=q^{-1}(0)$. By Theorem 3.1, Y is an M-summand in X since $\operatorname{span}(q)$ is an

L-summand in X^*. An M-projection has norm 1 and hence, by Lemma 1.4, Y has the E.SYM(2)-property. Thus Y is isometric to l_∞^{n-1} by the induction hypothesis. Since Y is an M-summand in X we have X isometric to l_∞^n.

LEMMA 3.7. *Let X be a G-space. If X has the* E.SYM(2)-*property then X is a C_σ-space.*

PROOF. Let $p \in$ w*-closure $(\partial_e X_1^*)$. Then by [4, theorem 7.10] there are $\alpha \in [-1,1]$ and $y \in \partial_e X_1^*$ such that $p = \alpha y$ and we have

$$g(p) = \alpha g(y) \qquad \text{for all } g \in X. \tag{3.2}$$

Since a G-space is an L_1-predual we have that span(y) is an L-summand in X^*. Hence there is an $f \in X$ such that $\|f\| = 1 = f(y)$. Let h be the center of $S = B(f, 2-\alpha) \cap B(-f, 2)$. A proof similar to the one we used to prove Lemma 2.3 will give that X^{**} has the E.SYM(2)-property. Let $\tilde{S} = B(f, 2-\alpha) \cap B(-f, 2)$ in X^{**}. Then $\tilde{S}$ has a center and by using arguments similar to those used to prove Lemma 2.5 we find that h is also a center of $\tilde{S}$. By the proof of Lemma 1.2 and by the uniqueness of centers (Lemma 1.1) we see that h must be the function in $X^{**} = C(T)$ defined by

$$2h = \max(f - 2 + \alpha, -f - 2) + \min(f + 2 - \alpha, -f + 2).$$

Now w*-closure $(\partial_e X_1^*) \subseteq T$ and hence

$$2h(y) = \max(-1+\alpha, -3) + \min(3-\alpha, 1) = (-1+\alpha) + 1 = \alpha,$$
$$2h(p) = \max(-2+2\alpha, -\alpha - 2) + \min(2, 2-\alpha) = |\alpha|.$$

But $h \in X$, hence by (3.2), $2h(p) = \alpha 2h(y)$ and $|\alpha| = \alpha^2$. Thus $\alpha = -1, 0$ or 1. This gives that w*-closure$(\partial_e X_1^*) \subseteq \partial_e X_1^* \cup \{0\}$, and by [3, theorem 13] X is a C_σ-space.

REMARK. It is possible to prove more than Lemma 3.7. Let X be an L_1-predual with the E.SYM(2)-property. Then by using a proof similar to the one above it is possible to prove that if $p \in$ w*-closure$(\partial_e X_1^*)$ then $p = 0$ or $\|p\| = 1$. Hence Lemma 3.7 will follow as a corollary.

THEOREM 3.8. *A real Banach space X is a C_σ-space if and only if X has the* E.SYM(3)-*property.*

PROOF. Use Lemma 1.2, Theorem 2.1 and Lemma 3.7.

COROLLARY 3.9 [10, theorem 3]. *The range of a norm*-1 *projection in a C_σ-space is a C_σ-space.*

PROOF. Lemma 1.4 and Theorem 3.8.

REFERENCES

1. E. Alfsen and E. Effros, *Structure in real Banach spaces*, Ann. of Math. (2) **96** (1972), 98–173.

2. E. Effros, *On a class of real Banach spaces*, Isr. J. Math. **9** (1971), 430–458.

3. H. Fakhoury, *Préduaux de L-espace, Notion de centre*, J. Funct. Anal. **9** (1972), 189–207.

4. Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Am. Math. Soc. **227** (1977), 1–62.

5. Å. Lima, *Intersection properties of balls in spaces of compact operators*, Ann. Inst. Fourier **28** (1978), 35–65.

6. Å. Lima, *On extreme operators on finite-dimensional Banach spaces whose unit balls are polytopes*, Ark. Mat. **19** (1981), 97–116.

7. Å. Lima, G. Olsen and U. Uttersrud, *Intersections of M-ideals and G-spaces*, Pac. J. Math., to appear.

8. J. Lindenstrauss, *Extensions of compact operators*, Mem. Am. Math. Soc. **48** (1964).

9. J. Lindenstrauss and P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Isr. J. Math. **7** (1969), 325–349.

10. J. Lindenstrauss and D. Wulbert, *On the classification of the Banach spaces whose duals are L_1-spaces*, J. Funct. Anal. **4** (1969), 332–349.

11. S. Mazur, *Über konvexe Mengen in linearen, normierten Räume*, Studia Math. **4** (1933), 70–84.

12. T. Rao, *Characterizations of some classes of L_1-preduals by the Alfsen–Effros structure topology*, Isr. J. Math. **42** (1982), 20–32.

13. N. Roy, *A characterization of square Banach spaces*, Isr. J. Math. **17** (1974), 142–148.

14. N. Roy, *An M-ideal characterization of G-spaces*, Pac. J. Math. **92** (1981), 151–160.

15. P. Taylor, *A characterization of G-spaces*, Isr. J. Math. **10** (1971), 131–134.

16. U. Uttersrud, *On M-ideals and the Alfsen–Effros structure topology*, Math. Scan. **43** (1978), 369–381.

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